Making a C_6-free graph C_4-free and bipartite

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Abstract

We show that every C_6-free graph G has a C_4-free, bipartite subgraph with at least $3e(G)/8$ edges. Our proof is probabilistic and uses a theorem of Füredi, Naor and Verstraëte on C_6-free graphs.

1 Introduction

For a graph G, let $e(G)$ denote the number of edges in G. We say G is H-free if it does not contain H as a subgraph. For a family of graphs $\mathcal{F}$, let $\mathrm{ex}(n, \mathcal{F})$ denote the maximum number of edges an n-vertex graph G can have such that G is F-free for all $F \in \mathcal{F}$.

Győri [2] proved that every bipartite, C_6-free graph contains a C_4-free subgraph with at least half as many edges. Extending this result, Kühn and Osthus [3] showed that every bipartite, C_{2k}-free graph has a C_4-free subgraph with at least $1/(k-1)$ of the original edges. In an extensive study of the Turán number $\mathrm{ex}(n, C_6)$, Füredi, Naor and Verstraëte [1] gave another generalization of Győri's result by showing (Theorem 3.1) that a C_6-free graph has a triangle-free, C_4-free subgraph with at least half as many edges.

Using any of these results combined with the well-known fact that every graph has a bipartite subgraph with at least half as many edges, it is easy to show that any C_6-free graph has a bipartite, C_4-free subgraph with at least $1/4$ the original edges. Improving the constant $1/4$ is the main focus of this paper.

In general, if we would like to make a C_6-free graph C_4-free and bipartite, we cannot hope to keep more than $2/5$ of its edges (consider many disjoint K_5's). We show that if c is the maximum constant such that every C_6-free graph G has a C_4-free subgraph on $c \cdot e(G)$ edges then $3/8 \le c \le 2/5$.

Theorem 1. *Let G be a C_6-free graph, then G contains a subgraph with at least $3e(G)/8$ edges which is both C_4-free and bipartite.*

The result can also be phrased in the language of Turán theory: If $\mathcal{C}$ denotes the set of all odd cycles, then $\mathrm{ex}(n, C_6) \le 8\,\mathrm{ex}(n, C_4, C_6, \mathcal{C})/3$.

Our proof is a probabilistic deletion procedure consisting of several steps. First we two-color the vertices, and then, focusing on specific edge-disjoint subgraphs, we delete certain edges given the outcome of the coloring. These edge-disjoint subgraphs are the maximal subgraphs obtained by pasting together edge-intersecting C_4's and were characterized by Füredi, Naor and Verstraëte. We use the following slightly weaker formulation of their theorem.

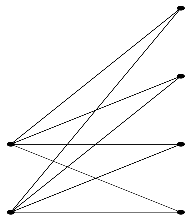
Theorem 2. *For a C_6-free graph G, let H denote the graph whose vertex set is the collection of C_4's in G and whose edge set represents edge-intersection. Each connected component of H corresponds to an induced subgraph of G of one of the following types:*

(0) the complete bipartite graph $K_{2,m}$ for some $m > 0$,

(1) a triangle xyz with α additional vertices adjacent to x and y, and β more vertices adjacent to x and z,

(2) a K_4 with $\gamma \ge 0$ paths of length 2 (outside the K_4) between two of the vertices,

(3) a K_5, K_5 minus an edge, or a K_5 minus two non-adjacent edges.



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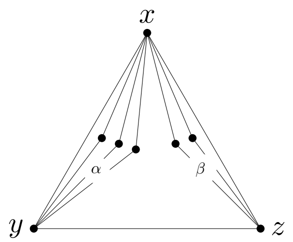

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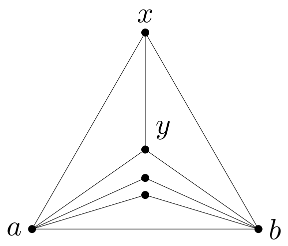

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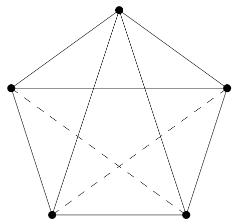

Type 3

2 Proof of Theorem 1

Independently at random, color all vertices in G red or blue with probability $1/2$ each. Deleting all monochromatic edges would yield a bipartite graph, but some C_4's may remain. Thus, given the random coloring we will deterministically delete additional edges in such a way that, upon deletion of monochromatic edges, at least $3e(G)/8$ edges remain in expectation, but all C_4's are deleted. Notice that after coloring, the C_4's which require further edge deletion are exactly the properly colored C_4's (those with no monochromatic edges).

For each component H of type 0, 1, 2, or 3 from Theorem 2 we will show that our vertex-coloring and subsequent edge-deletion procedure preserves at least $3e(H)/8$ edges in expectation. Since these components are edge-disjoint and cover all C_4's, we are then done by linearity of expectation.

Case(H is of type 0): First, suppose H is a component of type 0. That is, H is a complete bipartite graph $K_{2,t}$. Let x and y be the vertices in the first class, and $v_1, v_2, \ldots, v_t$ be the vertices in the second class. If x and y are opposite colors, then there are no properly colored C_4's, and the expected number of remaining edges is exactly $e(H)/2$.

Now, suppose that x and y are the same color, say red. If none of the v_i's are colored blue then we lose all edges in H. If exactly s, $s \geq 1$, of the v_i's are colored blue, then we must delete all but one of the edges emanating from x to the v_i's for otherwise we would have a properly colored C_4. Thus, exactly $s+1$ edges will remain in H. The probability that s of the v_i are blue is $\binom{t}{s}/2^t$. Let

N_0 be the random variable equal to the number of edges which remain in H, then

$$\begin{aligned}\mathbb{E}(N_0 \mid x \text{ and } y \text{ same color}) &= \frac{1}{2^t}0 + \sum_{s=1}^{t} \frac{\binom{t}{s}}{2^t}(s+1) \\ &= \frac{1}{2^t}\sum_{s=1}^{t}\binom{t}{s}s + \frac{1}{2^t}\sum_{s=1}^{t}\binom{t}{s} \\ &= \frac{1}{2^t}t2^{t-1} + \frac{1}{2^t}(2^t-1) \\ &\geq \frac{t}{2} + \frac{1}{2}.\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}(N_0) &= \frac{1}{2}\mathbb{E}(N_0 \mid x \text{ and } y \text{ opposite color}) + \frac{1}{2}\mathbb{E}(N_0 \mid x \text{ and } y \text{ same color}) \\ &\geq \frac{e(H)}{4} + \frac{t}{4} + \frac{1}{4} \\ &= \frac{3e(H)}{8} + \frac{1}{4}.\end{aligned}$$

Case(H is of type 1): Now, assume that H is of type 1. Let x, y, z be as in the figure. Assume that there are α vertices adjacent to x and y (excluding z), and β vertices adjacent to x and z (excluding y). Notice that $2\alpha + 2\beta = e(H) - 3$.

First suppose x, y and z are the same color. This subcase occurs with probability 1/4. The edges $\{x, y\}$, $\{x, z\}$ and $\{y, z\}$ are all monochromatic, so all properly colored C_4's are contained in one of two bipartite graphs, a $K_{2,\alpha}$ or a $K_{2,\beta}$. By the reasoning in the previous case we can preserve

$$\frac{\alpha}{2} + \frac{1}{2} + \frac{\beta}{2} + \frac{1}{2} = \frac{e(H)}{4} + \frac{3}{4}$$

edges in expectation.

Now, suppose x is one color and both y and z are the opposite color. This subcase also occurs with probability 1/4. We have that exactly two of the edges in the triangle formed by x, y and z are preserved as are half of the remaining edges. Thus, in total, we save $(e(H)-3)/2+2 = e(H)/2+1/2$ edges in expectation.

Next, assume that x and y are one color and z is the opposite color. This again happens with probability 1/4. In this subcase we must also consider C_4's through x, y, z and one of the α vertices other than z adjacent to x and y. To this end, we immediately delete the edge $\{y, z\}$. Now, only one edge remains on the triangle through x, y and z which is not monochromatic. Each of the β vertices is on one monochromatic edge and one properly colored edge. The vertices x, y and their α common neighbors again form a $K_{2,\alpha}$ which we handle as before, saving at least $\alpha/2 + 1/2$ edges in expectation. It follows that the expected total number of edges preserved in this subcase is $\alpha/2 + \beta + 3/2$.

The final subcase in which x and z are the same color and y is the opposite color is totally symmetric. In this case, the expected number of preserved edges is thus $\beta/2 + \alpha + 3/2$.

Let N_1 be the random variable equal to the number of edges conserved in H, then

$$\begin{aligned}\mathbb{E}(N_1) &= \frac{1}{4}(\frac{e(H)}{4}+\frac{3}{4})+\frac{1}{4}(\frac{e(H)}{2}+\frac{1}{2})+\frac{1}{4}(\frac{\alpha}{2}+\beta+\frac{3}{2})+\frac{1}{4}(\frac{\beta}{2}+\alpha+\frac{3}{2})\\ &= \frac{3}{16}e(H)+\frac{3}{8}(\alpha+\beta)+\frac{15}{16}\\ &= \frac{3}{16}e(H)+\frac{3}{16}(e(H)-3)+\frac{15}{16}\\ &> \frac{3}{8}e(H).\end{aligned}$$

Case(H is of type 2): We will condition first on whether a and b are the same color or opposite and then on whether x and y are the same color or opposite.

Suppose first that a and b are opposite colors. Then all C_4's lie in the subgraph induced by a, b, x and y. If x and y are the same color, no further edges need to be deleted. If x and y are opposite colors we must delete one additional edge. In either situation, exactly $e(H)/2$ edges are preserved.

Now, assume that a and b are the same color, say red. Consider the subcase when x and y are also red, then all properly colored C_4's must lie in a $K_{2,\gamma}$. By the reasoning we have used before, this implies that we can keep

$$\frac{e(H)-6}{4}+\frac{1}{2}=\frac{e(H)}{4}-1$$

edges in expectation.

If x and y are opposite colors, then 3 of the 6 edges in the K_4 defined by a, b, x and y remain. For each of the γ vertices which are blue we must delete an edge. Thus, we retain

$$3+\frac{e(H)-6}{4}=\frac{e(H)}{4}+\frac{3}{2}$$

edges in expectation.

Finally, if x and y are both blue, then delete the edge $\{a, x\}$. By the same reasoning as the preceding subcase we retain

$$3+\frac{e(H)-6}{4}=\frac{e(H)}{4}+\frac{3}{2}$$

edges in expectation. Letting N_2 be the random variable counting the number of preserved edges we have

$$\begin{aligned}\mathbb{E}(N_2) &= \frac{1}{2}\frac{e(H)}{2}+\frac{1}{8}(\frac{e(H)}{4}-1)+\frac{1}{4}(\frac{e(H)}{4}+\frac{3}{2})+\frac{1}{8}(\frac{e(H)}{4}+\frac{3}{2})\\ &= \frac{3}{8}e(H)+\frac{7}{16}\\ &\geq \frac{3}{8}e(H).\end{aligned}$$

Case(H is of type 3): H is either a K_5, a K_5 minus an edge or a K_5 minus two nonadjacent edges. First, suppose H is a K_5. There are three possibilities: all 5 vertices are the same color, there is a unique vertex of one color or there are two vertices of one color. These possibilities have probabilities $2/32, 10/32$ and $20/32$ respectively. In the first case we have 0 remaining edges and in the second we have 4. In the third we must delete 2 additional edges, again leaving a total of 4. Thus, if N_3 counts the expected number of edges remaining, we have

$$\mathbb{E}(N_3) = \frac{2}{32}0 + \frac{10}{32}4 + \frac{20}{32}4 = \frac{3}{8}e(H).$$

The analysis of K_5 minus one or two edges is similar.

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